GRAVIATIONAL REPULSION OF A NEUTRAL TEST PARTICLE IN KERR-DE SITTER SPACE TIME

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Abstract: In this work we have studied some features of geodesic motion of the neutral test particle in the equatorial plane of Kerr-de Sitter space time. The neutral test particle experiences attraction in between the outer black hole horizon and cosmological horizon, but it experiences repulsion as it approaches the outer black hole horizon and cosmological horizon.

Key words: Kerr-de Sitter spacetime; geodesics.

I. INTRODUCTION

The Kerr-de Sitter Space in Boyer Lindquest type of co-ordinate system is described by the line element (Khanal, 1985)

$$aS = g_{\mu\nu} ax \; ax \tag{1}$$

Where

(2)

Here

Δ_r = (r^2+a^2) -2Mr

$\rho^2 = r^2 + a^2 \cos^2\theta$
δ = $\mathrm{Sin}^2\theta$ (3)

$\Xi^2 = 1 + \dfrac{\Lambda a^2}{3}$

$\Delta_\theta = 1 + \dfrac{\Lambda a^2}{3} \mathrm{Cos}^2\theta$

Further, Λ is the cosmological constant, M is the mass of the black hole and a is the Kerr parameter defined as specific angular momentum of the black hole and given $a = \dfrac{L_z}{M}$.

The Kerr-de Sitter metric (2) is reduced into Kerr-metric when $\Lambda \to 0$ and also changes into schwarzchild metric when as well.

II. GEODETIC EQUATIONS

The Hamiltonian-Jacobi equation governing geodesic motion in a space-time with metric tensor $g^{\mu\upsilon}$ given by

$$2\frac{\partial S}{\partial \tau} = \frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} g^{\mu\nu} \tag{4}$$

where S denotes Hamiltons principle function. With $g^{\mu\upsilon}$ from Eq.(2), Eq. (4) becomes

$$2\frac{\partial S}{\partial \tau} = -\frac{\Xi^2}{\rho^2\Delta_r}\left[\left(r^2+a^2\right)\frac{\partial S}{\partial t} + a\left(\frac{\partial S}{\partial \phi}\right)\right]^2 + \frac{\Xi^2}{\rho^2\Delta_\theta \sin^2\theta}\left[a\sin^2\theta\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \phi}\right]^2 + \frac{\Delta_r}{\rho^2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{\Delta_\theta}{\rho^2}\left(\frac{\partial S}{\partial \theta}\right)^2 \tag{5}$$

Assuming that variables can be separated, we seek a solution of the form

$$S = \frac{1}{2}\delta_1\tau - Et + L_Z\phi + S_r(r) + S_\theta(\theta) \tag{6}$$

From equations (5) and (6),

$$\rho^2\delta_1 = -\frac{\Xi^2}{\Delta_r}\left[aL_Z - E\left(r^2+a^2\right)\right]^2 + \frac{\Xi^2}{\Delta_\theta\sin^2\theta}\left[L_Z - aE\sin^2\theta\right]^2 + \Delta_r\left(\frac{dS_r}{dr}\right)^2 + \Delta_\theta\left(\frac{dS_\theta}{d\theta}\right)^2 \tag{7}$$

Since $\Xi^2 = 1 - \dfrac{\Lambda a^2}{3}$ is always constant, and replacing $E\Xi$ by E and $L_z\Xi$ by L_z, Eq. (7) reduces to

$$\rho^2\delta_1 = -\frac{1}{\Delta_r}\left[aL_Z - E\left(r^2+a^2\right)\right]^2 + \frac{1}{\Delta_\theta}\left[L_Z - aE\right]^2 + \frac{\cos^2\theta}{\Delta_\theta}\left[L_Z \operatorname{cosec}^2\theta - a^2E^2\right] + \Delta_r\left(\frac{dS_r}{dr}\right)^2 + \Delta_\theta\left(\frac{dS_\theta}{d\theta}\right)^2 \tag{8}$$

By using separation of variables

$$\Delta_r\left(\frac{dS_r}{dr}\right)^2 = \frac{1}{\Delta_r}\left[\left(r^2+a^2\right)E - aL_Z\right]^2 - \left[K + r^2\delta_1\right] \tag{9}$$

and, $$\Delta_\theta\left(\frac{dS_\theta}{dr}\right)^2 = K - \frac{1}{\Delta_\theta}\left(L_Z - aE\right)^2 - \frac{\cos^2\theta}{\Delta_\theta}\left[L_Z \operatorname{cosec}^2\theta - a^2E^2\right] + a^2\cos^2\theta\delta_1 \tag{10}$$

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Where K is the separation constant.

Let us define the terms R and Θ by

(11)

(12)

Equation (6) becomes

(13)

The basic equations governing the motion can be deduced from the solution Eq. (13) for the principal function by the standard procedure of setting to zero the partial derivative of with respect to the different constants of the motion and in this instance. Thus we find that

$$\frac{\partial S}{\partial K} = \frac{1}{2}\int \frac{1}{\Delta_r \sqrt{R}} \frac{\partial R}{\partial K} dr + \frac{1}{2}\int \frac{1}{\Delta_\theta} \frac{1}{\sqrt{\Theta}} \frac{\partial \Theta}{\partial K} d\theta = 0$$

$$\int^r \frac{dr}{\sqrt{R}} = \int^\theta \frac{d\theta}{\sqrt{\Theta}} \qquad (14)$$

Similarly, we find

$$\tau = \int^r \frac{r^2}{\sqrt{R}} dr + a^2 \int^\theta \frac{\cos^2\theta}{\sqrt{\Theta}} d\theta \qquad (15)$$

S

$$t = \tau E + 2M \int^r r\left[r^2 E - a\left(L_Z - aE\right)\right] \frac{dr}{\Delta\sqrt{R}} \qquad (16)$$

and $\phi = a\int^r \left[\left(r^2 + a^2\right)E - aL_z\right] \frac{dr}{\Delta\sqrt{R}} + \int^\theta \left(L_z \operatorname{cosec}^2\theta - aE\right) \frac{d\theta}{\sqrt{\Theta}}$ (17)

The general Lagrangian is given by

$$2L = -\frac{1}{\rho^2\Xi^2}\left(\Delta_r - a^2\delta\Delta_\theta\right)\dot{t}^2 - \frac{4a\delta}{\rho^2\Xi^2}\left[\left(r^2 + a^2\right)\Delta_\theta - \Delta_r\right]\dot{t}\dot{\phi} + \frac{\rho^2}{\Delta_r}\dot{r}^2 + \frac{\rho^2}{\Delta_\theta}\dot{\theta}^2$$
$$+ \frac{\delta}{\rho^2\Xi^2}\left[\left(r^2 + a^2\right)^2\Delta_\theta - a^2\Delta_r \sin^2\theta\right]\dot{\phi}^2 \qquad (18)$$

Therefore,

$$\rho^2 \frac{dr}{d\tau} = \sqrt{R} \qquad (19)$$

$$\rho^2 \frac{d\theta}{d\tau} = \sqrt{\Theta} \qquad (20)$$

$$\rho^2 \frac{d\phi}{d\tau} = \frac{aP}{\Delta_r} + \frac{Y}{\Delta_\theta \sin^2\theta} \qquad (21)$$

$$\rho^2 \frac{dt}{d\tau} = \frac{B}{\Delta_r} \qquad (22)$$

where

$$\left.\begin{aligned} P &= E\left(r^2 + a^2\right) - L_Z a \\ B &= a\Delta_r Y + \left(r^2 + a^2\right)P \\ Y &= L_Z - aE\sin^2\theta \end{aligned}\right] \qquad (23)$$

From Eq. (19), we find

$$\frac{d^2 r}{dt^2} = \ddot{r} = \frac{\Delta_r}{B^3}\left[B\left((\Delta_r' + \frac{R'\Delta_r}{2}\right) - R\Delta B'\right]$$

Also from equation (21),

$$r\dot{\phi}^2 = r\left[\frac{aP\Delta_\theta + \Delta_r Y \sin^{-2}\theta}{\Delta_\theta B}\right]^2$$

The acceleration of a neutral test particle as observed by a distant observer is

$$\ddot{r} - r\dot{\phi}^2 = \frac{\Delta_r}{B^3}\left[B\left(R\Delta_r' + \frac{R'\Delta_r}{2}\right) - R\Delta_r B'\right] - r\left[\frac{aP\Delta_\theta + \Delta_r Y\sin^{-2}\theta}{\Delta_\theta B}\right]^2 \qquad (24)$$

In Eq.(24), the dots indicate differentiation with respect to t and primes indicate differentiation with respect to r. From Eq. (24) can be obtained as

$$A_e = M\left(\ddot{r} - r\dot{\phi}^2\right)$$
$$= \frac{D}{B_e^{\,3}}\left[B_e DG - 2\left(x^2 + A^2\right)\left(P_e^{\,2} - DZ\right)\left\{P_e H + DKx + P_e \frac{\Lambda M^2 x}{3}\left(2x^2 + A^2\right)\right\}\right] - x\left(\frac{AP_e + DX}{B_e}\right)^2 \qquad (25)$$

where

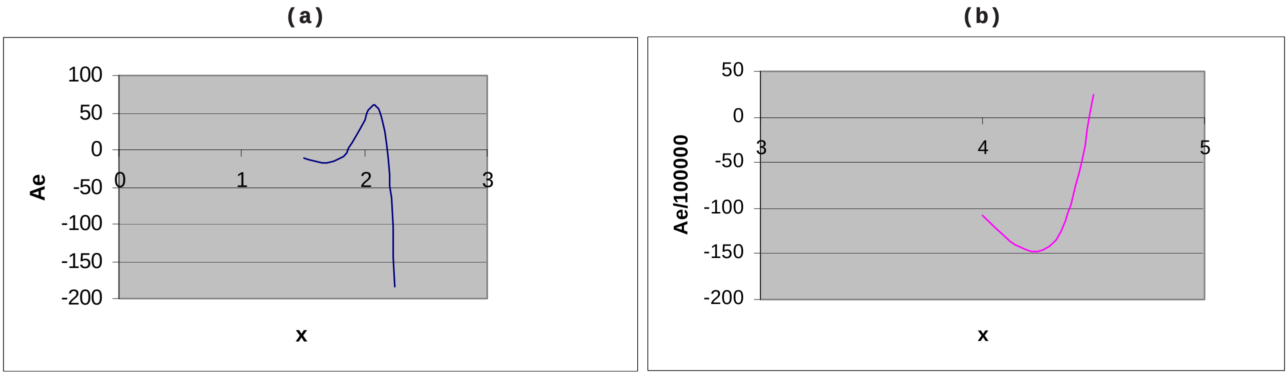
$$A_e = M\left(\ddot{r} - r\dot{\phi}^2\right), x = \frac{r}{M}, A = \frac{a}{M}, K = \frac{E}{\mu}, L = \frac{L_Z}{\mu M}$$
$$D = \frac{\Delta_r}{M^2}, \tau = \frac{t}{M} \, and D = x^2 - 2x + A^2$$
$$B_e = AXD + \left(x^2 + A^2\right)P_e$$
$$X = L - AK$$
$$Z = X^2 + x^2$$
$$G = 2KP_e x - (x-1)Z - Dx$$
$$P_e = K(x^2 + A^2) - LA$$
$$H = \frac{Dx}{x^2 + A^2} - x + 1$$

In Eq.(25), A_e is plotted against x for Kerr de Sitter source in equatorial plane $\left(\theta = \frac{\pi}{2}\right)$, is shown in figs. (1) and (2) for A=L=K=0.5 and X=0.25

In fig.1 we have plotted A_e against x as given by Eq. (25). We have taken A=L=K=0.5, X=0.25 and $\frac{\Lambda M^2}{3} = 0.01.$

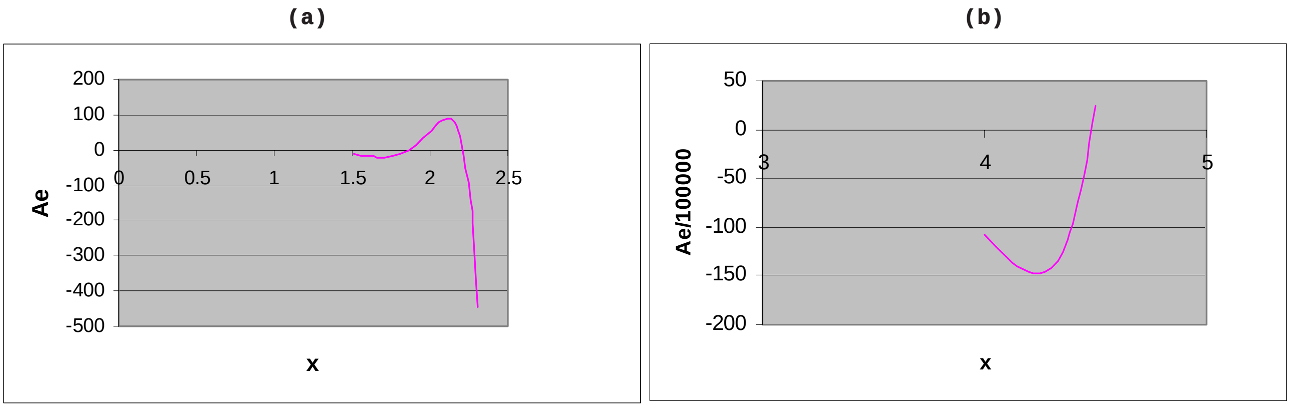
Fig. 2 is a similar plot for $\frac{\Lambda M^2}{3} = 0.001.$

It is seen from the plots that A_e which represents the gravitational "force" felt by the test particle becomes positive near both the outer black hole horizon and the cosmological horizon. The positivity of this "force" is the indication of repulsion. In particular, for the chosen values, the "force" becomes attractive for 2.15<x<4.5 when $\frac{\Lambda M^2}{3} = 0.01.$ These values of x can be compared to the



Plot of A_e Vs. x for a Kerr de Sitter source in equatorial plane $(\theta = \frac{\pi}{2})$ for A = L = K=0.5, X= 0.25 and $\frac{\Lambda M^2}{3}$ = 0.01

Fig. 1



Plot of A_e Vs. x for a Kerr de Sitter source in equatorial plane $(\theta = \frac{\pi}{2})$ for A = L = K=0.5, X= 0.25 and $\frac{\Lambda M^2}{3}$ = 0.01

Fig. 2

black hole horizon at x=1.95 and the cosmological horizon at x=8.79. Similarly for $\frac{\Lambda M^2}{3} = 0.001$, the "force" becomes attractive for 2.21<x<9.51; in this case, the outer black hole horizon is at x=1.87, and the cosmological horizon at x=30.57.

CONCLUSIONS

Hamilton-Jacobi method is used in Kerr-de Sitter space time to study geodetic motion of a neutral test particle. The "force" experienced by such a particle in the region between the outer black hole horizon and the cosmological horizon is given by A_e in Eq. (25). It is seen that such a test particle experiences repulsion when it approaches either of these horizons. The repulsion is indicated by A_e becoming positive. As it is quite certain that $\Lambda > 0$ in the universe, the space time geometry of any gravitating object will be of a de Sitter type. So it has become very important to study geodetic motion in such space time. A detailed study of such motion and the comparison with observation should give a good estimate of .

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