THE NULL GEODESICS IN KERR-DE SITTER SPACE TIME

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Abstract: Some properties of the angular null geodesics in Kerr-de Sitter space-time are investigated. These trajectories are unstable and oscillate about the equatorial plane but do not reach the singularity. These geodesics are found to be confined within a maximum angle about the equatorial plane. This maximum angle is dependent on the cosmological constant. So it is inferred that the thickness of accretion disks should be related to the value of cosmological constant.

Key Words: Kerr de Sitter space-time; geodesics.

A. INTRODUCTION

The delineation of the geodesics exhibits the essential features of the space-time. The method discovered by Carter [4] for the separability of the Hamilton-Jacobi equations is a very important aspect of the Kerr metric. The study led to the possibility of extracting energy from the Kerr black hole (The Penrose process). Such properties of the geodesics seem to be a common feature of all Petrov type-D matrix [2].

In the background of renewed interest in cosmological constant, after the publication of the results of Supernova Cosmology Project (SCP) and High Red-shift Supernova Team (HZT) [5, 6], a study of null geodesics in Kerr deSitter space-time has been carried out. Kerr-de Sitter space-time is characterized by the presence of the cosmological constant term in the solution of Einstein field equations.

B. METHOD OF HAMILTON-JACOBI

The Kerr-Newman-de Sitter space in a Boyer Lindquist type of coordinate system is described by the line element [3]

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{B.1}$$

where,

$$g^{\mu\nu} = \begin{pmatrix} \frac{\Xi^2}{\rho^2}\left[\frac{(r^2+a^2)^2}{\Delta_r} - \frac{a^2\sin^2\theta}{\Delta_\theta}\right] & 0 & 0 & \frac{a\Xi^2}{\rho^2}\left[\frac{(r^2+a^2)}{\Delta_r} - \frac{1}{\Delta_\theta}\right] \\ 0 & \frac{-\Delta_r}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{-\Delta_\theta}{\rho^2} & 0 \\ \frac{a\Xi^2}{\rho^2}\left[\frac{(r^2+a^2)}{\Delta_r} - \frac{1}{\Delta_\theta}\right] & 0 & 0 & \frac{-\Xi^2}{\rho^2\sin^2\theta}\left[\frac{1}{\Delta_\theta} - \frac{a^2\sin^2\theta}{\Delta_r}\right] \end{pmatrix} \tag{B.2}$$

where,

$$\left.\begin{aligned} \Delta_r &= \left(r^2+a^2\right)\left(1-\frac{\Lambda r^2}{3}\right) - 2Mr + Q^2 \\ \rho^2 &= r^2 + a^2\cos^2\theta \\ \Xi &= 1 + \frac{\Lambda a^2}{3} \\ \Delta_\theta &= 1 + \frac{\Lambda a^2\cos^2\theta}{3} \end{aligned}\right\} \tag{B.3}$$

where Λ is the cosmological constant, M is the mass of the black hole, Q is its charge and $a = L_z/M$ is the Kerr parameter defined as specific angular momentum of the black hole.

When $\Lambda=0$, the Kerr-deSitter metric reduces to the Kerr-Newman-metric. When $a=0$ and $\Lambda=0,$ it reduces to the Reissner-Nordstrom metric. With $Q=0$ and $\Lambda=0,$ it is reduced into Kerr metric; and also changes into Schwarzschild metric when $a=0$ as well. The extensive analysis of Kerr metric has been discussed by Carter [7].

The Hamilton-Jacobi equation governing the geodesic motion in a space-time with metric tensor $g^{\mu\nu}$ is given by

$$2\frac{\partial S}{\partial \tau} = \frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}} g^{\mu\nu} \tag{B.4}$$

where S denotes the principal function.

Assuming that the variables can be separated, we seek a solution of (B.4) of the form:

$$S = \frac{1}{2}\delta\tau - Et + L_z\Phi + S_r(r) + S_\theta(\theta) \tag{B.5}$$

Quite generally, geodesic motion in a stationary axisymmetric space-time will allow two integrals of motion: the energy and the angular momentum about the axis of symmetry. Besides, the norm of the four velocity will also be conserved by virtue of its parallel propagation. These three conservation laws will not, in general, suffice to reduce the problem of solving the equations of geodesic motion to one involving quadratures only [2]. Carter's discovery of a further conserved quantity "K" has contributed for such reduction.

With substitutions and little bit of algebra, we find the solution for S is given by:

$$S = \frac{1}{2}\delta\tau - Et + L_z\Phi + \int^{r}\frac{\sqrt{R(r)}}{\Delta_r}dr + \int^{\theta}\frac{\sqrt{\Theta(\theta)}}{\Delta_\theta}d\theta \tag{B.6}$$

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(C.8)

C.2. The r-motion

The equations determining unstable orbits of constant radius are given by

(C.9)

$$\frac{\partial R}{\partial r} = 4r[r^2 + a^2 - a\xi] - \eta\,\frac{\partial \Delta_r}{\partial r} = 0 \qquad \text{(C.10)}$$

Under the constraint of the principal null-congruence, we find

$$\eta = \frac{4r^2[(r^2+a^2)(1-\frac{\Lambda r^2}{3})-2Mr]}{[r(1-\frac{\Lambda a^2}{3})-\frac{2\Lambda r^3}{3}-M]^2} \qquad \text{(C.11)}$$

and, $$\xi = \frac{r\left(r^2+a^2\right)\left(1+\frac{\Lambda a^2}{3}\right)+M\left(a^2-3r^2\right)}{a\left[M-r\left(1-\frac{\Lambda a^2}{3}\right)+\frac{2\Lambda r^3}{3}\right]} \qquad \text{(C.12)}$$

These equations determine, parametrically, the critical locus (ξ_s,). The plot of and is shown in Fig. 1 for different values of .

Starting with R given by (C.9), it can be reduced to

(C.13)

[illegible] $= \mu/\mu_+$

where, $k_1 = 1 + \frac{16\Lambda r_s^{\,2}\Delta_s}{3\left(\partial\Delta_s/\partial s\right)^2}$ and $\Delta_s = \left(r_s^{\,2}+a^2\right)\left(1-\frac{\Lambda r_s^{\,2}}{3}\right)-2Mr_s$.

with substitution, $(r - r_s)^{-1} = x$, Equation (C.13) can be reduced to

$$R = \frac{k_1}{x^4}(1 + 4r_s\,x + k_2\,x^2)$$

where, $k_2 = 4r_s^{\,2} - \frac{16M\,r_s\,\Delta_s}{k_1\left(\partial\Delta_s/\partial s\right)^2}$

$$\Rightarrow \int\frac{dr}{\sqrt{R}} = \int\frac{dx}{\sqrt{k_1(1+4r_s\,x+k_2\,x^2)}}$$

$$= \frac{1}{\sqrt{k_1 k_2}}\,ln[2\sqrt{(1+4r_s\,x+k_2\,x^2)} + \frac{2(2r_s+k_2x)}{\sqrt{k_2}}] \qquad \text{(C.14)}$$

Thus,

$$\int\frac{d\mu}{\sqrt{\Theta_\mu}} = \frac{1}{\sqrt{k_1 k_2}}\,ln[2\sqrt{(1+4r_s\,x+k_2\,x^2)} + \frac{2(2r_s+k_2x)}{\sqrt{k_2}}] \qquad \text{(C.15)}$$

Where the constants and variables have their usual meanings. Equation (C.15) determines the projection of the orbits on the (r, θ) plane.

Trajectories derived from equation (C.15) for different values of are illustrated in Fig.3.

D. RESULTS AND DISCUSSIONS

A study of the unstable circular null geodesics in the Kerr de Sitter space-time has been carried out in this work, so that the effect of on the motion of mass-less particles in the field of rotating gravitating body could be studied.

The effect of on and , which represents the energy and angular momentum, has been studied.

Fig. 2: The plot of /10^2 for corresponding values of r_s with the Kerr parameter a = 0.8 for different values of Λ. **(a)** =0.11; and **(b)** =0.015625. In addition to the constraints that >0, we have to fulfill the condition that >0.

Fig. 3: Critical null-geodesics in the (r, θ) plane for r_s = 3; **(a)** , and =178.23. **(b)** $\Lambda = 0.015625$, =39.308 and =31.783. These orbits do not reach the singularity. The unit of length along the coordinate axes is M, and the chosen Kerr parameter is 0.8.

As the test particle rotates around Kerr-de Sitter black hole in a circular orbit, it oscillates symmetrically about the equatorial plane as shown in Fig. 3.a and Fig. 3.b. The effect of on maximum value to which particle can swing ( ) has been studied and shown in Fig. 6. From this, it is seen that adds a constraint on the value of : as increase, decreases. This indicate that the thickness of the accretion disk around the black hole should depend significantly on the value of .

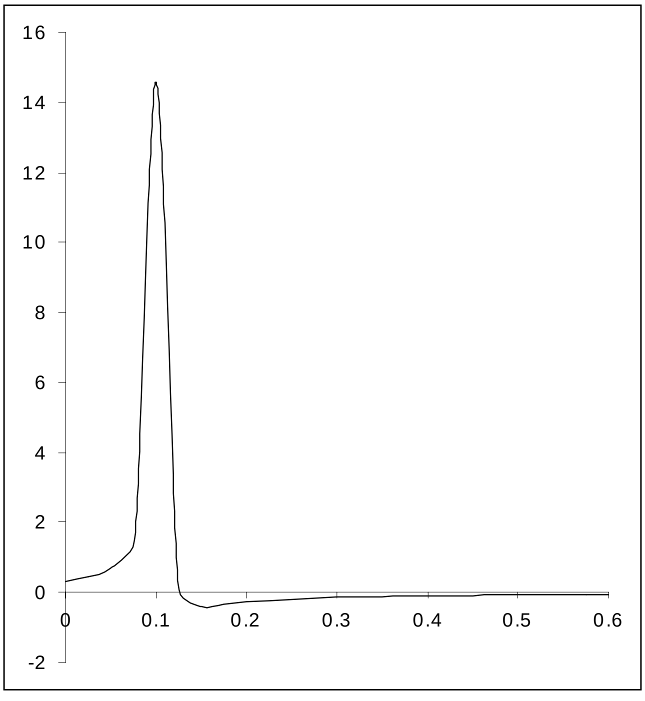

Fig. 4: Graph showing variation of /10^2 with .

Fig.5: Graph showing variation of /10^2 with Λ.

Fig. 6: Plot of with for r_s = 3.0 and a = 0.8.

This may provide a method to estimate the value of the Cosmological Constant.

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